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LETTERS TO THE EDITOR

VIBRATION ANALYSIS OF PLATES WITH CURVILINEAR QUADRILATERAL PLANFORMS BY DQM USING BLENDING FUNCTIONS

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1. INTRODUCTION

The differential quadrature method (DQM) is a numerical technique introduced in the early 1970s by Bellman and his associates [1, 2] for the solution of initial and boundary value problems. The method has been applied to a variety of physical problems including transport processes, static and dynamic structural mechanics, and hydrodynamic lubrication. An exhaustive list of the literature on the DQM and its developmental history may be found in a recent survey paper by the present authors [3]. The general effectiveness of the DQM and its analytical simplicity have made it a possible alternative to the well-known finite difference and finite element methods.

Due to its very basis, the DQM may be applied very easily to regions having boundaries aligned to the reference co-ordinate axes. Thus, the domains considered in early applications of the DQM have been line domains for one-dimensional and axisymmetric problems and rectangular domains for plane problems. Recently, the DQM was applied to the analysis of vibration and buckling of skew and rhombic plates using oblique co-ordinate axes [4]. More recently, the DQM was applied to curvilinear quadrilateral domains by using the technique of natural-to-Cartesian geometric mapping [5]. To do this, the quadrature rules were reformulated using the serendipity-family interpolation functions. However, with such functions, the mapping cannot be exact except at the nodal points of the boundary; indeed, this matter requires very careful consideration of the boundary conditions.

The difficulties of mapping boundaries of curvilinear quadrilateral domains can be eliminated by using blending functions which permit exact mapping [6]. These functions have been used in finite element analysis [7–9], but they are introduced for the first time to the DQM in the present work.

In the following, the procedure for constructing the quadrature rules for curvilinear quadrilateral domains via geometric mapping by blending functions is

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described. Next, the quadrature formulation of the vibration problem of thin plates is considered for application of the proposed methodology. The free vibration frequencies calculated for sectorial plates with simply supported and clamped edges are compared with available data.

2. MAPPING AND QUADRATURE RULES

Let the field domain of interest be a curvilinear quadrilateral region in the Cartesian x–y plane; see Figure 1(a). The geometric mapping of this domain may be achieved from a square parent domain, $-1 \leqslant \xi \leqslant 1$, $-1 \leqslant \eta \leqslant 1$ located in the natural ξ–η plane; see Figure 1(b). This mapping is carried by use of blending functions [6–9]

$$\begin{aligned} s = &(\tfrac{1}{2})\{(1-\eta)\bar{s}_1(\xi) + (1+\xi)\bar{s}_2(\eta) + (1+\eta)\bar{s}_3(\xi) + (1-\xi)\bar{s}_4(\eta)\} \\ &- (\tfrac{1}{4})\{(1-\xi)(1-\eta)s_1 + (1+\xi)(1-\eta)s_2 + (1+\xi)(1+\eta)s_3 \\ &+ (1-\xi)(1+\eta)s_4\}, \end{aligned} \qquad (1)$$

where $s = x, y$. As shown in Figure 1(a), $\bar{x}_i(\xi)$, $\bar{x}_i(\eta)$, $\bar{y}_i(\xi)$, $\bar{y}_i(\eta)$ are the parametric equations of the curvilinear boundaries and x_i, y_i are the Cartesian co-ordinates of the corner points of the quadrilateral region. It is noted that the geometric mapping of the boundaries of the quadrilateral domain by the blending functions of equation (1) is exact.

Now, consider the quadrature rules for the first order derivatives of a function f with respect to the ξ, η co-ordinates in the parent square domain of Figure 1(b). At a sampling point (ξ_i, η_j) of the quadrature grid in the square domain (Figure 2) these rules may be expressed as [1, 3]

$$\left.\frac{\partial f}{\partial \xi}\right|_{\xi_i, \eta_j} = \sum_{k=1}^{N_\xi} P_{ik} f_{kj}, \qquad \left.\frac{\partial f}{\partial \eta}\right|_{\xi_i, \eta_j} = \sum_{l=1}^{N_\eta} Q_{jl} f_{il}, \qquad (2)$$

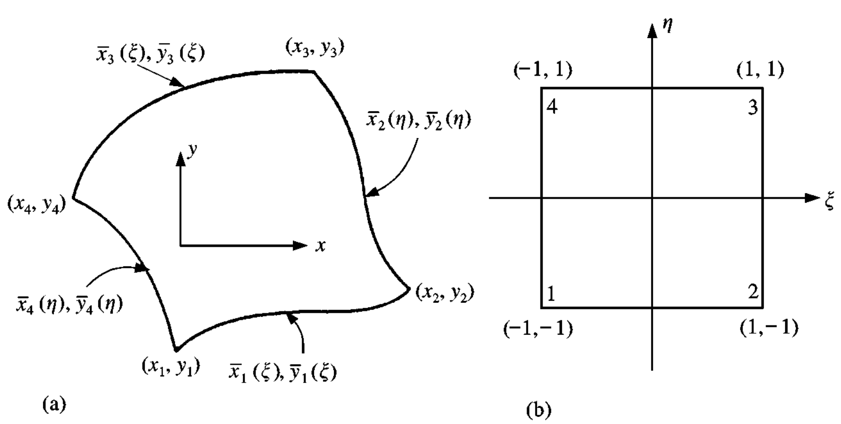

Figure 1. Domains: (a) curvilinear quadrilateral region in the Cartesian x–y plane; (b) a square parent domain in the natural ξ–η plane.

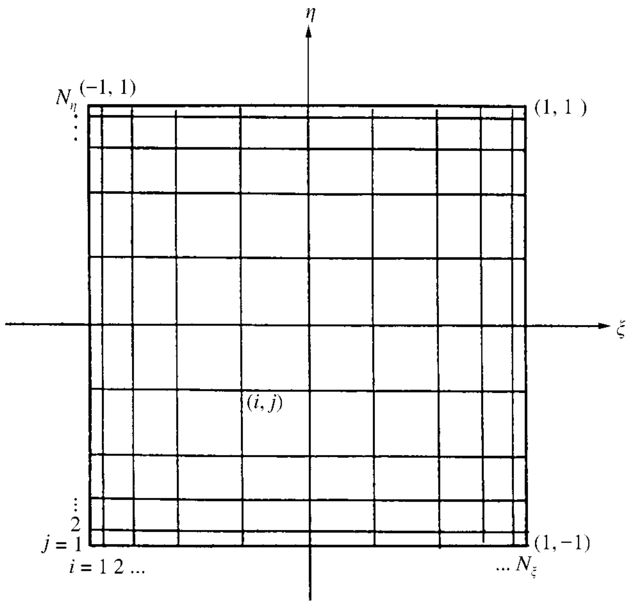

Figure 2. A quadrature grid.

where $f_{ij} = f(\xi_i, \eta_j)$, P_{ik} and Q_{jl} are the weighting coefficients, and N_ξ, N_η are the number of sampling points in the ξ and η directions respectively.

By the chain rule of differentiation,

$$\frac{\partial f}{\partial x} = \frac{1}{|J|}\left(\frac{\partial y}{\partial \eta}\frac{\partial f}{\partial \xi} - \frac{\partial y}{\partial \xi}\frac{\partial f}{\partial \eta}\right), \qquad \frac{\partial f}{\partial y} = \frac{1}{|J|}\left(-\frac{\partial x}{\partial \eta}\frac{\partial f}{\partial \xi} + \frac{\partial x}{\partial \xi}\frac{\partial f}{\partial \eta}\right), \tag{3}$$

where $|J|$ is the determinant of the Jacobian $J = \partial(x, y)/\partial(\xi, \eta)$.

Using equations (2) in equations (3), one obtains the quadrature rules for the first order derivatives of $f = f(x, y)$ at a point $x_{ij} = x(\xi_i, \eta_j)$, $y_{ij} = y(\xi_i, \eta_j)$ of the mapped domain as

$$\left(\frac{\partial f}{\partial x}\right)_{ij} = \frac{1}{|J_{ij}|}\left[\left(\frac{\partial y}{\partial \eta}\right)_{ij}\sum_{k=1}^{N_\xi} P_{ik} f_{kj} - \left(\frac{\partial y}{\partial \xi}\right)_{ij}\sum_{l=1}^{N_\eta} Q_{jl} f_{il}\right] \text{ or } \left(\frac{\partial f}{\partial x}\right)_m = \sum_{n=1}^{N_{\xi\eta}} A_{mn}^{(1)} f_n, \tag{4}$$

$$\left(\frac{\partial f}{\partial y}\right)_{ij} = \frac{1}{|J_{ij}|}\left[-\left(\frac{\partial x}{\partial \eta}\right)_{ij}\sum_{k=1}^{N_\xi} P_{ik} f_{kj} + \left(\frac{\partial x}{\partial \xi}\right)_{ij}\sum_{l=1}^{N_\eta} Q_{jl} f_{il}\right] \text{ or } \left(\frac{\partial f}{\partial y}\right)_m = \sum_{n=1}^{N_{\xi\eta}} B_{mn}^{(1)} f_n, \tag{5}$$

where $m,\ n = (i-1)N_\eta + j,\ j = 1, 2, \ldots, N_\eta,\ i = 1, 2, \ldots, N_\xi,\ N_{\xi\eta} = N_\xi \times N_\eta$, and $A_{mn}^{(1)}$ and $B_{mn}^{(1)}$ are the weighting coefficients of the first order derivatives.

The general quadratures rules in the mapped region may be written as

$$\left(\frac{\partial^r f}{\partial x^r}\right)_m = \sum_{n=1}^{N_{\xi\eta}} A_{mn}^{(r)} f_n, \left(\frac{\partial^s f}{\partial y^s}\right)_m = \sum_{n=1}^{N_{\xi\eta}} B_{mn}^{(s)} f_n, \left(\frac{\partial^{r+s} f}{\partial x^r \partial y^s}\right)_m = \sum_{n=1}^{N_{\xi\eta}} C_{mn}^{(rs)} f_n \tag{6}$$

in which the weighting coefficients may be obtained from the recurrence relationships [3, 5]

$$[A^{(r)}] = [A^{(1)}][A^{(r-1)}],\ [B^{(s)}] = [B^{(1)}][B^{(s-1)}] \quad (r, s \geqslant 2),$$
$$[C^{(rs)}] = [A^{(r)}][B^{(s)}] \quad (r, s \geqslant 1). \tag{7}$$

3. THE THIN PLATE VIBRATION PROBLEM

The governing differential equation of the mode function $W = W(X, Y)$ of free harmonic vibration of a thin isotropic plate in a dimensionless form is

$$W_{,XXXX} + 2W_{,XXYY} + W_{,YYYY} - \Omega^2 W = 0, \tag{8}$$

where $X, Y = x/a, y/a$ are dimensionless Cartesian co-ordinates in the plane of the midplane of the plate, $\Omega = \omega a^2(\rho h/D)^{1/2}$ is the dimensionless frequency, ω is a natural frequency, a, h, and D are, respectively, a characteristic in-plane dimension, thickness, and flexural rigidity of the plate; and ρ is the density of the plate material. Also, $W_{,XXXX} = \partial^4 W/\partial X^4$, etc.

The boundary conditions of a clamped edge (zero deflection and zero normal rotation) and a simply supported edge (zero deflection and zero normal moment) are

$$W = 0,\ W_{,X}\cos\theta + W_{,Y}\sin\theta = 0,$$
$$(\cos^2\theta + \nu\sin^2\theta)W_{,XX} + (\sin^2\theta + \nu\cos^2\theta)W_{,YY}$$
$$+ \{2(1-\nu)\cos\theta\sin\theta\}W_{,XY} = 0, \tag{9}$$

where θ is the angle between the normal to the plate boundary and the x-axis.

Using equations (6), one obtains the following quadrature analogs of equation (8) and (9):

$$\sum_{n=1}^{N_{\xi\eta}} \{A_{mn}^{(4)} + 2C_{mn}^{(22)} + B_{mn}^{(4)}\} W_n = \Omega^2 W_m, \tag{10}$$

$$W_m = 0,\ \sum_{n=1}^{N_{\xi\eta}} \{(\cos\theta_m)A_{mn}^{(1)} + (\sin\theta_m)B_{mn}^{(1)}\} W_n = 0;$$

$$\sum_{n=1}^{N_{\xi\eta}} \{(\cos^2\theta_m + \nu\sin^2\theta_m)A_{mn}^{(2)} + (\sin^2\theta_m + \nu\cos^2\theta_m)B_{mn}^{(2)}$$
$$+ 2(1-\nu)(\cos\theta_m\sin\theta_m)C_{mn}^{(11)}\} W_n = 0. \tag{11}$$

4. NUMERICAL RESULTS

The foregoing DQM formulation was employed for calculating free vibration frequencies of sectorial plates with various types of boundary conditions. The plate geometry is shown in Figure 3. Here, in Table 1, sample results are given for two types of plates, namely, a SSSS plate, simply supported (S) all over the four edges, and a SCSC plate, simply supported on the radial edges and clamped (C) on the

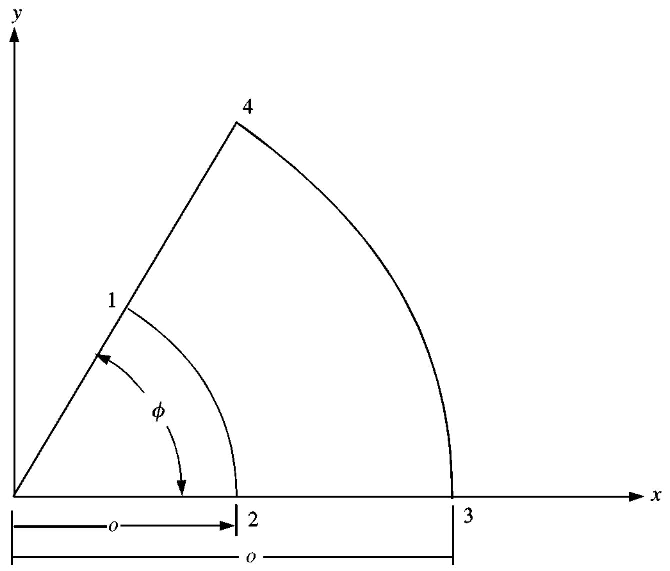

Figure 3. A concentric, circular, sectorial region.

TABLE 1

The first four free vibration frequencies of sectorial plates ($v = 0{\cdot}3$)

	Modes							
$N_\xi = N_\theta$	1	2	3	4	1	2	3	4
	SSSS plate				*SCSC plate*			
Exact analytical solution [10]								
—	68·379	150·98	189·60	278·39	107·57	178·82	269·49	305·84
Eight-term orthogonal-polynomial Rayleigh–Ritz solution [10]								
	68·379	150·98	189·60	278·39	107·57	178·82	269·49	305·84
DQM solution with blending functions								
11	68·379	150·98	189·60	278·17	107·57	178·82	269·51	305·63
12	68·379	150·98	189·60	278·42	107·57	178·82	269·49	305·88
13	68·379	150·98	189·60	278·38	107·57	178·82	269·49	305·84
14	68·379	150·89	189·60	278·39	107·57	178·82	269·49	305·84
15	68·379	150·98	189·60	278·39	107·57	178·82	269·49	305·84
DQM solution with cubic serendipity interpolation functions [5]								
11	68·364	150·94	189·62	279·31	107·57	178·79	269·52	305·58
12	68·378	150·91	189·57	278·38	107·57	178·79	269·49	305·84
13	68·376	150·95	189·60	278·12	107·57	178·79	269·50	305·80
14	68·379	150·95	189·60	278·34	107·57	178·79	269·50	305·80
15	68·378	150·95	189·60	278·35	107·57	178·79	269·50	305·80

circumferential edges. For each plate $a/b = 2$, $\phi = 45°$. Table 1 also includes the exact analytical results obtained by Kim and Dickinson [10] using the methodology of Ramakrishnan and Kunukkaseril [11], as well as an eight-term orthogonal polynomial Rayleigh–Ritz solution [10] and the results from reference [5] in which a DQM solution was obtained by using cubic serendipity-family interpolation functions.

Comparing the results obtained using the various methods, it is noted that all three sets of numerical-method results for both types of plates agree very closely with the exact results, mostly to five significant figures and in some instances to only four. Also, it is apparent that blending-function mapping leads to an improvement in the convergence of the DQM.

5. CONCLUDING REMARKS

An improved methodology for use with the differential quadrature method, based on the use of blending functions for the mapping, has been introduced and successfully validated for the case of free vibration of sectorial plates with two different sets of boundary conditions.

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